Creation of chaos for a mean-field model

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Pierre Del Moral* & Julian Tugaut†

Abstract

The article deals with the propagation of chaos for a system of interacting particles. Under suitable assumptions, if the system at time $t = 0$ is chaotic (that is to say the particles are independent), this chaos propagates as the number of particles goes to infinity. Here, we deal with a case in which the system at time $t = 0$ is not chaotic and we show under easily checked assumptions that the system becomes chaotic as the number of particles goes to infinity together with the time. This yields the first result of this type for mean field particle diffusion models.

Key words and phrases: Interacting particles system ; Propagation of chaos ; McKean-Vlasov models ; Nonlinear diffusions

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1 Introduction

In the current work, we show that under suitable assumptions, there is creation of chaos then propagation of this chaos for a mean-field system of particles without assuming that the initial random variables are independent. In other words, the particles become independent as the time t goes to infinity if the number of particles is large. About propagation of chaos, we refer the reader to [Szn91, Mél96].

The paper is organized as follows. First, we present the model of interacting particles system that we consider. Then, we explain the idea of the propagation of chaos and we give some classical results about it. Next paragraph is about the hydrodynamical limit, that is the so-called McKean-Vlasov diffusion, which can be seen as the probabilistic interpretation of a non-linear partial differential equation, the granular media equation. Then, we give a classical coupling result between the particles of the mean-field model and the McKean-Vlasov diffusion. After giving the precise assumptions, we provide the main results of the article. The following section deals with the creation of chaos for the hydrodynamical

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limit of the mean-field system of particles, that is not the McKean-Vlasov diffusion since there is no chaos at time $t = 0$. Finally, we prove the main results about the creation of chaos for the mean-field model.

1.1 Mean-field model

Let V and F be two potentials on $\mathbb{R}$. The precise hypotheses are given subsequently. For instance, we assume that both V and F are $\mathcal{C}^2$-continuous and convex at infinity. If the number of particles, N, is finite, the system corresponds to a classical diffusion in $\mathbb{R}^N$. In other words, we consider N diffusions in $\mathbb{R}$ with N independent one-dimensional Wiener processes and non-independent initial random variables. We add a friction term, that is the gradient of the external potential V. Moreover, we assume that each particle is under the influence of the global behaviour of the particles system. Here, we assume that each particle is attracted by any other one and that the interaction depends only on the distance between the particles. Thus, the mean-field system that we consider here is a random dynamical system in which each particle $X^{i,N}$ satisfies the stochastic differential equation:

$$X_t^{i,N} = X_0^i + \sigma B_t^i - \int_0^t \nabla V\left(X_s^{i,N}\right) ds - \frac{1}{N}\int_0^t \sum_{j=1}^N \nabla F\left(X_s^{i,N} - X_s^{j,N}\right) ds\,, \quad (1)$$

where the N Brownian motions $\left(B_t^i\right)_{t\in\mathbb{R}_+}$ are independent and the initial random variables $\left(X_0^i\right)_i$ follow the law μ_0.

Let us remark that the gradient in Equation (1) permits roughly speaking to locate the particles in a compact and to have good long-time properties.

Let us stress that we do not assume any independence between the particles $X_0^1, \cdots, X_0^N$. In fact, we take $X_0^1 = X_0^2 = \cdots = X_0^N =: X_0$. Nevertheless, the Brownian motions are independent from the random variable X_0. In this paper, we show that the particles $X^{i,N}$ and $X^{j,N}$ become independent as t and N are large.

Here, the function V is called the confining potential. Indeed, it attracts each diffusion to its minimizers. The potential F is the so-called interacting potential. Due to the assumptions on the interaction, the function $x \mapsto \nabla F(x)$ is radial.

By

$$\eta_t^N := \frac{1}{N}\left(\delta_{X_t^{1,N}} + \cdots + \delta_{X_t^{N,N}}\right)\,,$$

we denote the empirical measure of the particles system. We observe that the Equation (1) can be rewritten like so:

$$X_t^{i,N} = X_0^i + \sigma B_t^i - \int_0^t \nabla V\left(X_s^{i,N}\right) ds - \int_0^t \nabla F * \eta_s^N\left(X_s^{i,N}\right) ds\,.$$

And, by using Itô formula, we obtain

$$\frac{d}{dt}\mathbb{E}\left\{\int_{\mathbb{R}^d} f\eta_t^N\right\} = \mathbb{E}\left\{\frac{\sigma^2}{2}\int_{\mathbb{R}^d}\Delta f\eta_t^N - \int_{\mathbb{R}^d}\left\langle\nabla f\,;\,\nabla V + \nabla F * \eta_t^N\right\rangle\eta_t^N\right\}\,,$$

for any smooth function with compact support f from $\mathbb{R}$ to $\mathbb{R}$. Let us notice that if a family of deterministic measures $\{\eta_t\,;\,t\geq 0\}$ was satisfying the previous equation, it would be a solution of the so-called granular media equation,

$$\frac{\partial}{\partial t}\mu_t = \mathrm{div}\left\{\frac{\sigma^2}{2}\nabla\mu_t + (\nabla V + \nabla F * \mu_t)\,\mu_t\right\}\,. \tag{2}$$

Heuristically, if the family of random measures $\left\{\eta_t^N\,;\,0\leq t\leq T\right\}$ converges to a family of deterministic measures $\{\eta_t\,;\,0\leq t\leq T\}$, this deterministic family satisfies the non-linear partial differential equation (2). Since we take N arbitrarily large, it motivates to focus on this family of measures.

1.2 Propagation of chaos

In this paragraph, let us momentarily assume the following hypothesis.

$$X_0^i \quad \text{and} \quad X_0^j \quad \text{are independent for any } i\neq j\,. \tag{C_0}$$

This hypothesis is equivalent to

$$\mathcal{L}\left(X_0^1,\cdots,X_0^N\right) = \mu_0^{\otimes N}\,. \tag{C_0'}$$

We say that the system at time $t=0$ is chaotic. This terminology is due to Boltzmann. We can observe that $X_t^{i,N}$ and $X_t^{j,N}$ are not independent if t is positive. Indeed, the particle $X^{j,N}$ intervenes in the drift of Equation (1). So, we do not have the property:

$$\mathcal{L}\left(X_t^{1,N},\cdots,X_t^{N,N}\right) = \left(\mu_t^N\right)^{\otimes N}\,, \tag{C_t'}$$

with $\mu_t^N := \mathcal{L}\left(X_t^{1,N}\right) = \cdots = \mathcal{L}\left(X_t^{N,N}\right)$. One says that there is propagation of chaos if the system becomes chaotic on any finite time interval, as N goes to infinity. In other words, propagation of chaos holds if for any $T>0$, we have

$$\lim_{N\to\infty}\left(\mathcal{L}\left(X_t^{1,N},\cdots,X_t^{k,N}\right)\right)_{0\leq t\leq T} = \left((\eta_t)^{\otimes k}\right)_{0\leq t\leq T}\,, \tag{3}$$

η_t being a probability measure. Here, k is any integer. Another way to see the propagation of chaos is the following. Let k be any integer and let $f_1,\cdots,f_k$ be k Lipschitz-continuous functions. Then, we have:

$$\mathbb{E}\left\{\prod_{i=1}^k f_i\left(X_t^{i,N}\right)\right\} - \prod_{i=1}^k\mathbb{E}\left\{f_i\left(X_t^{i,N}\right)\right\}\longrightarrow 0\,, \tag{4}$$

as N goes to infinity. This result holds for any $t \in [0;T]$. This means that a finite number of particles become independent all together.

Under simple hypotheses (potential V is Lipschitz-continuous or is convex at infinity), see for instance [Szn91] and [Mél96], the mean-field system of particles (1) satisfies the propagation of chaos. Let us point out that the whole system of particle is not chaotic. Indeed, even with propagation of chaos, we do not have

$$\lim_{N\to\infty}\left(\mathcal{L}\left(X_t^{1,N},\cdots,X_t^{N,N}\right)\right)_{0\leq t\leq T}=\left((\eta_t)^{\otimes\infty}\right)_{0\leq t\leq T}.$$

Ben Arous and Zeitouni go further than (4) by proving

$$\mathbb{E}\left\{\prod_{i=1}^{k(N)}f_i\left(X_t^{i,N}\right)\right\}-\prod_{i=1}^{k(N)}\mathbb{E}\left\{f_i\left(X_t^{i,N}\right)\right\}\longrightarrow 0\,,$$

where $k(N)$ is an integer such that $k(N)$ goes to infinity as N goes to infinity. The assumption on $k(N)$ is that it is negligible to N: $\frac{k(N)}{N}\longrightarrow 0$ as N goes to infinity.

1.3 McKean-Vlasov diffusions

Let us remind the reader that the particles are exchangeables. Consequently, we have:

$$\mathcal{L}\left(X_t^{i,N}\right)=\mathcal{L}\left(X_t^{j,N}\right)=:\mu_t^N\,. \tag{5}$$

Propagation of chaos means that particles become independent so that intuitively, by using the strong law of large number, the empirical measure η_t^N converges as N goes to infinity:

$$\eta_t^N=\frac{1}{N}\sum_{i=1}^N\delta_{X_t^{i,N}}\longrightarrow\eta_t=:\mathcal{L}\left(X_t^{1,\infty}\right)\,. \tag{6}$$

This convergence would hold for any $t>0$. However, since the drift in (1) is $\nabla V+\nabla F*\eta_t^N$, it converges toward $\nabla V+\nabla F*\eta_t$ as N goes to infinity. Equation (1) intuitively becomes

$$X_t^{i,\infty}=X_0^i+\sigma B_t^i-\int_0^t\nabla V\left(X_s^{i,\infty}\right)ds-\int_0^t\nabla F*\mathcal{L}\left(X_s^{i,\infty}\right)\left(X_s^{i,\infty}\right)ds\,.$$

This equation corresponds to the hydrodynamical limit of the mean-field system. More rigorously, we consider the N following diffusions, the so-called McKean-Vlasov diffusions:

$$X_t^i=X_0^i+\sigma B_t^i-\int_0^t\nabla V\left(X_s^i\right)ds-\int_0^t\nabla F*\eta_s\left(X_s^i\right)ds\,, \tag{7}$$

with $\eta_s:=\mathcal{L}\left(X_t^1\right)=\cdots=\mathcal{L}\left(X_t^N\right)$. These McKean-Vlasov diffusions correspond to the probabilistic interpretation of the granular media equation (2).

Indeed, due to [McK66, McK67], the measure of probability $\mathcal{L}\left(X_t^i\right)$ is absolutely continuous with respect to the Lebesgue measure for $t>0$. By u_t, we denote its density. Then, $(u_t)_{t>0}$ satisfies the following partial differential equation:

$$\frac{\partial}{\partial t}u_t=\nabla.\left\{\frac{\sigma^2}{2}\nabla u_t+\left(\nabla V+\nabla F*u_t\right)u_t\right\}\,. \tag{8}$$

The existence and the uniqueness of a strong solution on $\mathbb{R}_+$ for equation (7) has been proved in [HIP08] (Theorem 2.13). The asymptotic behaviour of the law has been studied in [CGM08, BRV98, BCCP98, CMV03, BGG13] (for the convex case) and in [Tug13a, Tug13b] in the non-convex case by using the results in [HT10a, HT10b, HT12] about the non-uniqueness of the invariant probabilities and their small-noise behaviour.

1.4 Coupling results

The McKean-Vlasov diffusion X^1 can be seen as the limit as N goes to infinity of the first particle in the mean-field model, $X^{1,N}$. Indeed, under Hypothesis (C_0), we have the following limit:

$$\lim_{n\to\infty}\sup_{0\leq t\leq T}\mathbb{E}\left\{\left|\left|X_t^{i,N}-X_t^i\right|\right|^2\right\}=0\,. \tag{9}$$

This limit holds for any integer i. We can go further. Indeed, propagation of chaos if system is chaotic at time $t=0$ is equivalent to

$$\lim_{n\to\infty}\mathbb{E}\left\{\sup_{0\leq t\leq T}\left|\left|X_t^{i,N}-X_t^i\right|\right|^2\right\}=0\,, \tag{10}$$

Remark 1.1. *Let us point out that the two diffusions are defined with the same Brownian motion.*

The two limits hold if the potential F is convex and if V is identically equal to 0, see [BRTV98]. Cattiaux, Guillin and Malrieu go further by proving a uniform propagation of chaos if V and F are convex, not necessarily uniformly strictly convex:

$$\lim_{n\to\infty}\sup_{t\geq 0}\mathbb{E}\left\{\left|\left|X_t^{i,N}-X_t^i\right|\right|^2\right\}=0\,, \tag{11}$$

still if the system is chaotic at time $t=0$.

In [Tug12a], a result which combines uniformity on time and on space has been obtained:

$$\lim_{\substack{N\to\infty\\ \sigma\to 0}}\mathbb{E}\left\{\sup_{t\in\left[0;e^{\frac{2H}{\sigma^2}}\right]}\left|\left|X_t^{i,N}-X_t^i\right|\right|^2\right\}=0\,, \tag{12}$$

where V and F are both convex. Here $H>0$ does depend on V and F.

Finally, more recently, see [DMT13], we have obtained a uniform propagation of chaos result between the laws of $X_t^{1,N}$ and of X^1 for the Wasserstein distance:

$$\lim_{N\to\infty}\mathbb{W}_2\left(\mathcal{L}\left(X_t^{1,N}\right)\,;\,\mathcal{L}\left(X_t^1\right)\right)=0\,. \tag{13}$$

This result holds without any hypothesis of convexity on V nor F. Nevertheless, we require the noise to be sufficiently large. To do so, we use the rate of convergence of the McKean-Vlasov diffusion to the unique invariant probability.

We use a similar argument in this paper: the convergence in long-time of the law $\mathcal{L}\left(X_t^1\right)$ to one of the invariant probabilities. However, we can not use the exact arguments in [DMT13].

Remark 1.2. *In* (13)*, the two diffusions are not necessarily defined with the same Brownian motion.*

Here, we do not have chaos at initial time. Consequently, we do not make the coupling with a McKean-Vlasov diffusion but with another type of diffusion which corresponds to the hydrodynamical limit of the system of interacting particles. We begin by defining the McKean-Vlasov diffusion Y^{x_0} like so:

$$\begin{cases} Y_t^{x_0}=x_0+\sigma B_t-\int_0^t\nabla V\left(Y_s^{x_0}\right)ds-\int_0^t\nabla F*\mu_s\left(Y_s^{x_0}\right)ds \\ \mu_t=\mathcal{L}\left(Y_t^{x_0}\right)\end{cases}.$$

The diffusion that we use here is the diffusion Y defined by $Y_t=Y_t^{X_0}$. Indeed, the drift of this diffusion is - as t is small - close to $x\mapsto\nabla F\left(x-X_0\right)$. This is exactly the drift - still for small t - of the first particle in the system of mean-field interacting particles.

Subsequently, we consider the diffusions X^i defined like Y with the Brownian motions B^i.

1.5 Hypotheses

We now present the exact assumptions of the paper on the potentials V and F and on the initial measure of probability, μ_0. First, we give the hypotheses on the confining potential V.

Assumption (A-1): *V is a $\mathcal{C}^2$-continuous function.*

Assumption (A-2): *For all $\lambda>0$, there exists $R_\lambda>0$ such that $\nabla^2V(x)>\lambda$, for any $||x||\geq R_\lambda$.*

The aim of this hypothesis is to confine the diffusion so that there is no explosion. We can observe that under assumptions (A-1) and (A-2), there exist a convex potential V_0 and $\theta\in\mathbb{R}$ such that $V(x)=V_0(x)-\frac{\theta}{2}||x||^2$.

Assumption (A-3) *The gradient ∇V is slowly increasing: there exist $m\in\mathbb{N}^*$ and $C>0$ such that $||\nabla V(x)||\leq C\left(1+||x||^{2m-1}\right)$, for all $x\in\mathbb{R}^d$.*

This assumption together with the same kind of assumptions on F ensure us that there is a global solution if some moments of μ_0 are finite.

Let us present now the assumptions on the interacting potential F:

Assumption (A-4): *There exists a function G from $\mathbb{R}_+$ to $\mathbb{R}$ such that $F(x) = G(||x||)$.*

In other words, the function F is radial which means that the interaction between two particles is only related to their distance.

Assumption (A-5): *G is an even polynomial function such that $\deg(G) =: 2n \geq 2$ and $G(0) = 0$.*

This hypothesis is used for simplifying the study of the invariant probabilities. Indeed, see [HT10a, HT10b, HT12, Tug13c, Tug12b], the research of an invariant probability is equivalent to a fixed-point problem in infinite dimension. Nevertheless, under Assumption (A-5), it reduces to a fixed-point problem in finite dimension.

Assumption (A-6): *And, $\lim_{r\to+\infty} G''(r) = +\infty$.*

Immediately, we deduce the existence of an even polynomial and convex function G_0 such that $F(x) = G_0(||x||) - \frac{\alpha}{2}||x||^2$, α being a real constant.

We also need hypotheses on the initial measure μ_0:

Assumption (A-7) *the $8q^2$th moment of the measure μ_0 is finite with $q := \max\{m, n\}$.*

Under Assumptions (A-1)–(A-7), Equation (7) admits a unique strong solution. Indeed, the assumptions of Theorem 2.13 in [HIP08] are satisfied: ∇V and ∇F are locally Lipschitz, G' is odd, ∇F grows polynomially, ∇V is continuously differentiable and there exists a compact $\mathcal{K}$ such that $\nabla^2 V$ is uniformly positive on $\mathcal{K}^c$. Moreover, we have the following inequality for a positive M_0:

$$\max_{1\leq j\leq 8q^2} \sup_{t\in\mathbb{R}_+} \mathbb{E}\left[||X_t||^j\right] \leq M_0\,. \tag{14}$$

In the following, we use the long-time convergence of the measure μ_t toward an invariant probability μ and the rate of convergence. We need a complementary hypothesis:

Assumption (A-8) *Diffusion* (7) *admits a unique invariant probability μ. Moreover, there exists $C_\sigma > 0$ such that*

$$\mathbb{W}_2(\mu_t; \mu) \leq e^{-C_\sigma t}\mathbb{W}_2(\mu_0; \mu)$$

for any initial measure μ_0 which is absolutely continuous with respect to the Lebesgue measure and with finite entropy.

We know that this property is satisfied under simple assumptions:

- If both confining potential V and interacting potential F are convex, there is a unique invariant probability. And, we have the asked inequality. See [BGG13].
- In the general dimension case, let us assume that there exist a strictly convex function Θ such that $\Theta(y) > \Theta(0) = 0$ for all $y \in \mathbb{R}^d$ and $p \in \mathbb{N}$

such that the following limit holds for any $y \in \mathbb{R}^d$: $\lim_{r \to +\infty} \frac{V(ry)}{r^{2p}} = \Theta(y)$. If $p > n$ and if σ is large enough, Theorem 1.7 in [DMT13] ensures us that there is a unique invariant probability. And, we have the asked inequality.

Under the Hypotheses (A-1)–(A-8), the probability measure μ_t converges exponentially for Wasserstein distance to the unique invariant probability μ as soon as the initial measure μ_0 is absolutely continuous with respect to the Lebesgue measure and with finite entropy.
Let us briefly justify why we can extend this inequality by starting from a Dirac measure: $\mu_0 = \delta_{x_0}$ with $x_0 \in \mathbb{R}^d$. To do so, we consider a sequence of probability measures with finite entropy $\left(\mu_0^{(n)}\right)_{n \geq 1}$ which converges for the Wasserstein distance to μ_0. By μ_t (respectively $\mu_t^{(n)}$), we denote the law at time t of the McKean-Vlasov diffusion starting from the law μ_0 (respectively the law $\mu_0^{(n)}$). Then, we have :

$$\mathbb{W}_2\left(\mu_t; \mu\right) \leq \mathbb{W}_2\left(\mu_t; \mu_t^{(n)}\right) + \mathbb{W}_2\left(\mu_t^{(n)}; \mu\right) .$$

By applying the inequality in Hypothesis (A-8) to $\mu_t^{(n)}$, we get

$$\mathbb{W}_2\left(\mu_t; \mu\right) \leq \mathbb{W}_2\left(\mu_t; \mu_t^{(n)}\right) + e^{-C_\sigma t}\mathbb{W}_2\left(\mu_0^{(n)}; \mu\right) .$$

By making a coupling, one can easily show that the quantity $\mathbb{W}_2\left(\mu_t; \mu_t^{(n)}\right)$ converges to 0. Finally, since $\mathbb{W}_2\left(\mu_0^{(n)}; \mu_0\right)$ goes to 0 as n tends to infinity, we obtain the formula

$$\mathbb{W}_2\left(\mu_t; \mu\right) \leq e^{-C_\sigma t}\mathbb{W}_2\left(\mu_0; \mu\right) .$$

Consequently, μ_t goes to μ as t goes to infinity.

One says that the set of Assumptions (A) is satisfied if Hypotheses (A-1)–(A-8) are assumed.
Finally, we remind the reader that

$$X_0^1 = X_0^2 = \cdots = X_0^N =: X_0 .$$

1.6 Main results of the paper

We now end the introduction by presenting the main results of the current work.

Theorem 1.3. *Let f_1 and f_2 be two Lipschitz-continuous functions. Under the set of Assumptions (A), for all $\epsilon > 0$, for all $T > 0$, there exist $t_0(\epsilon)$ and $N_0(\epsilon)$ such that*

$$\sup_{N \geq N_0(\epsilon)} \sup_{t \in [t_0(\epsilon); t_0(\epsilon)+T]} \left| \mathrm{Cov}\left[f_1\left(X_t^{1,N}\right) \; ; \; f_2\left(X_t^{2,N}\right)\right]\right| \leq \epsilon .$$

We can remark that a small covariance implies a phenomenon of chaos. Consequently, we have creation of chaos after time $t_0(\epsilon)$. And, there is propagation of this chaos on an interval of length T.

Theorem 1.4. *Let f_1 and f_2 be two Lipschitz-continuous functions. Under the set of Assumptions (A), if moreover, V and F are convex then, for all $\epsilon > 0$, there exist $t_0(\epsilon)$ and $N_0(\epsilon)$ such that*

$$\sup_{N\geq N_0(\epsilon)}\sup_{t\geq t_0(\epsilon)}\left|\mathrm{Cov}\left[f_1\left(X_t^{1,N}\right)\,;\,f_2\left(X_t^{2,N}\right)\right]\right|\leq\epsilon\,.$$

Here, we have a uniform propagation of chaos after the creation of chaos. Let us remark that, in Theorem 1.3 and in Theorem 1.4, we consider only two particles but we have the same result with any integer k.

Theorem 1.5. *Let $f_1,\cdots,f_k$ be k Lipschitz-continuous functions. Under the set of Assumptions (A), for all $\epsilon > 0$, for all $T > 0$, there exist $t_0(\epsilon)$ and $N_0(\epsilon)$ such that*

$$\sup_{N\geq N_0(\epsilon)}\sup_{t\in[t_0(\epsilon);t_0(\epsilon)+T]}\left|\mathbb{E}\left\{\prod_{i=1}^k f_i\left(X_t^{i,N}\right)\right\}-\prod_{i=1}^k\mathbb{E}\left\{f_i\left(X_t^{i,N}\right)\right\}\right|\leq\epsilon\,.$$

If moreover V and F are convex, we have

$$\sup_{N\geq N_0(\epsilon)}\sup_{t\geq t_0(\epsilon)}\left|\mathbb{E}\left\{\prod_{i=1}^k f_i\left(X_t^{i,N}\right)\right\}-\prod_{i=1}^k\mathbb{E}\left\{f_i\left(X_t^{i,N}\right)\right\}\right|\leq\epsilon\,.$$

We also have results about the empirical measure of the system. In case of chaos, this measure is close to a measure of the form $\nu^{\otimes N}$.

Theorem 1.6. *Let f_1 and f_2 be two Lipschitz-continuous functions. Under the set of Assumptions (A), for all $\epsilon > 0$, for all $T > 0$, there exist $t_1(\epsilon)$ and $N_1(\epsilon)$ such that*

$$\sup_{N\geq N_1(\epsilon)}\sup_{t\in[t_1(\epsilon);t_1(\epsilon)+T]}\left|\mathrm{Cov}\left[\eta_t^N(f_1)\,;\,\eta_t^N(f_2)\right]\right|\leq\epsilon$$

with $\eta_t^N(f):=\frac{1}{N}\sum_{i=1}^N f_i\left(X_t^{i,N}\right)$. If, moreover, both V and F are convex, we have

$$\sup_{N\geq N_1(\epsilon)}\sup_{t\geq t_1(\epsilon)}\left|\mathrm{Cov}\left[\eta_t^N(f_1)\,;\,\eta_t^N(f_2)\right]\right|\leq\epsilon$$

We conjecture that, bu using the same technics, one should be able to obtain creation of chaos for more general mean-field models providing that the hydrodynamical limit is stable in long-time.

2 Creation of chaos in the hydrodynamical limit

In the following, we look at the quantity $\mathbb{E}\left\{f_1\left(X_t^1\right)f_2\left(X_t^2\right)\right\}$. We remark that

$$\begin{aligned}\mathbb{E}\left\{f_1\left(X_t^1\right)f_2\left(X_t^2\right)\right\}&=\mathbb{E}\left\{f_1\left(X_t^1\right)\mathbb{E}\left[f_2\left(X_t^2\right)\ \mid\ X_0^1,\left(B_s^1\right)_{0\le s\le t}\right]\right\}\\&=\mathbb{E}\left\{f_1\left(X_t^1\right)\mathbb{E}\left[f_2\left(X_t^2\right)\ \mid\ X_0^1\right]\right\}\,.\end{aligned}$$

Consequently, to study the long-time behaviour of $\mathbb{E}\left\{f_1\left(X_t^1\right)f_2\left(X_t^2\right)\right\}$ requires to study $\mathbb{E}\left[f_2\left(X_t^2\right)\ \mid\ X_0^1\right]$. Since X_0^1 and X_0^2 are not independent, we do not have $\mathbb{E}\left[f_2\left(X_t^2\right)\ \mid\ X_0^1\right]=\mathbb{E}\left[f_2\left(X_t^2\right)\right]$.

According to previous results, see [BCCP98, BGG13, BRV98, CGM08, CMV03] for the convex case and [Tug13a, Tug13b] for the general case, we know that the measure μ_t converges weakly to μ as t goes to infinity, under the assumptions of the article. However, we do not know anything about the convergence of $\mathbb{E}\left[f_2\left(X_t^2\right)\ \mid\ X_0^1\right]$ as t goes to infinity. This is the purpose of next proposition.

Proposition 2.1. *Let f be a Lipschitz function from $\mathbb{R}$ to itself. Under the set of assumptions (A), we have:*

$$\mathbb{E}\left\{f_2\left(X_t^2\right)\ \mid X_0^1\right\}\longrightarrow\int_{\mathbb{R}}f_2(x)\mu(dx)\,,\tag{15}$$

and the convergence holds almost surely, as t goes to infinity.

Proof. For any x_0, we can write

$$\mathbb{E}\left\{f\left(X_t^2\right)\ \mid\ X_0\right\}\mathbb{1}_{\{X_0=x_0\}}=\mathbb{E}\left\{f\left(Y_t^{x_0}\right)\right\}\mathbb{1}_{\{X_0=x_0\}}$$

Consequently, for any random variable X which follows the law μ, we have

$$\begin{aligned}&\left|\mathbb{E}\left\{f\left(X_t^2\right)\ \mid\ X_0\right\}\mathbb{1}_{\{X_0=x_0\}}-\mathbb{E}\left\{f\left(X\right)\right\}\mathbb{1}_{\{X_0=x_0\}}\right|\\&\le\mathbb{E}\left\{\left|f\left(Y_t^{x_0}\right)-f\left(X\right)\right|\right\}\mathbb{1}_{\{X_0=x_0\}}\le C\mathbb{E}\left\{\left|Y_t^{x_0}-X\right|\right\}\mathbb{1}_{\{X_0=x_0\}}\,.\end{aligned}$$

By taking X which minimizes, we find

$$\begin{aligned}&\left|\mathbb{E}\left\{f\left(X_t^2\right)\ \mid\ X_0\right\}\mathbb{1}_{\{X_0=x_0\}}-\int f(x)\mu(dx)\mathbb{1}_{\{X_0=x_0\}}\right|\\&\le C\mathbb{W}_2\left(\mathcal{L}\left(Y_t^{x_0}\right);\mu\right)\mathbb{1}_{\{X_0=x_0\}}\\&\le Ce^{-C_\sigma t}\mathbb{W}_2\left(\delta_{x_0};\mu\right)\mathbb{1}_{\{X_0=x_0\}}\\&\le Ce^{-C_\sigma t}\sqrt{\int_{\mathbb{R}}\left(x-x_0\right)^2\mu(dx)\mathbb{1}_{\{X_0=x_0\}}}\,.\end{aligned}$$

This tends to 0 as t goes to infinity which achieves the proof.

□

Let us remark that we have obtained better: the convergence is exponential.

Now, we can prove that, as time t goes to infinity, the law of the couple (X_t^1,X_t^2) becomes the tensorial product of the marginal laws.

Proposition 2.2. *Let f_1 and f_2 be two Lipschitz functions from $\mathbb{R}$ to itself. Under the set of assumptions (A), we have:*

$$\mathbb{E}\left\{f_1\left(X_t^1\right) f_2\left(X_t^2\right)\right\} \longrightarrow \left(\int_{\mathbb{R}} f_1(x)\mu(dx)\right)\left(\int_{\mathbb{R}} f_2(x)\mu(dx)\right). \qquad (16)$$

The convergence holds as t goes to infinity.

Proof. We observe that

$$\begin{aligned}\mathbb{E}\left\{f_1\left(X_t^1\right) f_2\left(X_t^2\right)\right\} =&\mathbb{E}\left\{f_1\left(X_t^1\right)\mathbb{E}\left[f_2\left(X_t^2\right) \mid X_0^1, B^1\right]\right\}\\ =&\mathbb{E}\left\{f_1\left(X_t^1\right)\mathbb{E}\left[f_2\left(X_t^2\right) \mid X_0^1\right]\right\}\\ =&\left(\int_{\mathbb{R}} f_2(x)\mu(dx)\right)\mathbb{E}\left\{f_1\left(X_t^1\right)\right\}\\ &+\mathbb{E}\left\{f_1\left(X_t^1\right) A_t\right\},\end{aligned}$$

with

$$A_t := \mathbb{E}\left\{f_2\left(X_t^2\right) \mid X_0^1\right\} - \int_{\mathbb{R}} f_2(x)\mu(dx).$$

The limit in (15) gives us the convergence almost surely of the random variable A_t to 0 as t goes to infinity.

Furthermore, since f_2 is Lipschitz-continuous and according to the boundedness of the moments of X_t^2, we have the following inequality:

$$\begin{aligned}\mathbb{E}\left(||A_t||^2\right) &\leq 2\mathbb{E}\left[\left|\left|f_2\left(X_t^2\right)\right|\right|^2\right] + 2\left(\int_{\mathbb{R}} f_2(x)\mu(dx)\right)^2\\ &\leq C\left\{1+\mathbb{E}\left[\left|\left|X_t^2\right|\right|^2\right]\right\}\\ &\leq C_\sigma.\end{aligned}$$

By Lebesgue theorem, we deduce the following limit:

$$\lim_{t\to\infty}\mathbb{E}\left\{f_1\left(X_t^1\right) A_t\right\} = 0.$$

Moreover, due to the set of assumptions on the initial random variable, we have the following convergence as t goes to infinity:

$$\mathbb{E}\left[f_1\left(X_t^1\right)\right] \longrightarrow \int_{\mathbb{R}} f_1(x)\mu(dx).$$

This achieves the proof. □

Let us remark that the convergence is exponential.

In fact, we could have obtained a more general result by proceeding similarly.

Remark 2.3. *Let $f_1, \cdots, f_k$ be k functions Lipschitz-continuous. Then, under the hypotheses of Proposition 2.2, we have the convergence almost surely of*

$$\mathbb{E}\left\{\prod_{i=1}^{k} f_i\left(X_t^i\right)\right\}$$

toward

$$\prod_{i=1}^{k}\int_{\mathbb{R}} f_i(x)\mu(dx)\,.$$

By observing that $\prod_{i=1}^{k}\mathbb{E}\left[f_i\left(X_t^i\right)\right]$ converges to $\prod_{i=1}^{k}\int_{\mathbb{R}} f_i(x)\mu(dx)$, we immediately obtain the following theorem.

Theorem 2.4. *Let f_1 and f_2 be two Lipschitz functions from $\mathbb{R}$ to itself. Under the set of assumptions (A), we have:*

$$\mathrm{Cov}\left(f_1\left(X_t^1\right)\,;\,f_2\left(X_t^2\right)\right)\longrightarrow 0\,, \tag{17}$$

as t goes to infinity.

More generally, let any $k\geq 2$ and let $f_1, \cdots, f_k$ be k Lipschitz-continuous functions. Thus, we have the following convergence almost surely as t goes to infinity:

$$\mathbb{E}\left\{\prod_{i=1}^{k} f_i\left(X_t^i\right)\right\}-\prod_{i=1}^{k}\mathbb{E}\left\{f_i\left(X_t^i\right)\right\}\longrightarrow 0\,.$$

Let us point out that to obtain this result, we only use the convergence in long-time. We do not need to know anything about the rate of convergence.

However, we know that this convergence is exponential.

3 Creation of chaos in the mean-field system

We first provide a coupling result.

Proposition 3.1. *We assume that V, F and μ_0 satisfy the set of Hypotheses (A). Let X_0 be a random variable which follows the law μ_0. Then, for any $T>0$, we have the following inequality:*

$$\sup_{t\in[0;T]}\mathbb{E}\left\{\left|\left|X_t^i-X_t^{i,N}\right|\right|^2\right\}\leq\frac{C(\mu_0)}{N}\exp\left[2CT\right]\,, \tag{18}$$

where $C(\mu_0)$ is a positive function of $\int_{\mathbb{R}^d}||x||^{8q^2}\mu_0(dx)$ and C is a positive constant.

Proof. By μ_t, we denote the solution of the granular media equation starting from δ_{X_0}. By definition, for any $1 \leq i \leq N$, we have

$$\begin{aligned}
X_t^{i,N} - X_t^i = &- \int_0^t \left\{ \nabla V(X_s^{i,N}) - \nabla V(X_s^i) \right\} ds \\
&- \int_0^t \left\{ \frac{1}{N} \sum_{j=1}^N \nabla F(X_s^{i,N} - X_s^{j,N}) - \nabla F * \mu_s(X_s^i) \right\} ds \,.
\end{aligned}$$

We apply Itô formula to $X_t^{i,N} - X_t^i$ with the function $x \mapsto ||x||^2$. By introducing the notation $\xi_i(t) := \left|\left|X_t^{i,N} - X_t^i\right|\right|^2$, we obtain

$$\begin{aligned}
&d\xi_i(t) = -2\Delta_1(i,t)dt - \frac{2}{N}\Delta_2(i,t)dt \\
\text{with} \quad &\Delta_1(i,t) := \left\langle X_t^{i,N} - X_t^i \,;\, \nabla V(X_t^{i,N}) - \nabla V\left(X_t^i\right) \right\rangle \\
\text{and} \quad &\Delta_2(i,t) := \left\langle X_t^{i,N} - X_t^i \,;\, \sum_{j=1}^N \left[\nabla F(X_t^{i,N} - X_t^{j,N}) - \nabla F * \mu_t\left(X_t^i\right) \right] \right\rangle \,.
\end{aligned}$$

By taking the sum on the integer i running between 1 and N, we get

$$\begin{aligned}
&d\sum_{i=1}^N \xi_i(t) = -2\Delta_1(t)dt - \frac{2}{N} \sum_{i=1}^N \sum_{j=1}^N \Big(\Delta_2(i,j,t) + \Delta_3(i,j,t) \Big) dt \\
\text{with} \quad &\Delta_1(t) := \sum_{i=1}^N \Delta_1(i,t) \;, \\
&\Delta_2(i,j,t) := \left\langle \nabla F(X_t^{i,N} - X_t^{j,N}) - \nabla F(X_t^i - X_t^j) \,;\, X_t^{i,N} - X_t^i \right\rangle \\
\text{and} \quad &\Delta_3(i,j,t) := \left\langle \nabla F(X_t^i - X_t^j) - \nabla F * \mu_t\left(X_t^i\right) \,;\, X_t^{i,N} - X_t^i \right\rangle \,.
\end{aligned}$$

According to the definition of the function F_0 in Hypothesis (A-6), it is convex. This implies $\langle x - y \,;\, \nabla F_0(x-y) \rangle \geq 0$ for any $x, y \in \mathbb{R}^d$. This inequality yields

$$\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N \left(\Delta_2(i,j,t) + \Delta_2(j,i,t) \right) \geq -4\alpha \sum_{i=1}^N \left|\left| X_t^{i,N} - X_t^i \right|\right|^2 \,.$$

Indeed, we have

$$
\begin{aligned}
&\Delta_2(i,j,t)+\Delta_2(j,i,t)\\
=&\left\langle \nabla F(X_t^{i,N}-X_t^{j,N})-\nabla F(X_t^i-X_t^j)\,;\,\left(X_t^{i,N}-X_t^i\right)-\left(X_t^{j,N}-X_t^j\right)\right\rangle\\
=&\left\langle \nabla F_0(X_t^{i,N}-X_t^{j,N})-\nabla F_0(X_t^i-X_t^j)\,;\,\left(X_t^{i,N}-X_t^i\right)-\left(X_t^{j,N}-X_t^j\right)\right\rangle\\
&-\alpha\left\langle \nabla(X_t^{i,N}-X_t^{j,N})-(X_t^i-X_t^j)\,;\,\left(X_t^{i,N}-X_t^i\right)-\left(X_t^{j,N}-X_t^j\right)\right\rangle\\
=&\left\langle \nabla F_0(X_t^{i,N}-X_t^{j,N})-\nabla F_0(X_t^i-X_t^j)\,;\,\left(X_t^{i,N}-X_t^{j,N}\right)-\left(X_t^i-X_t^j\right)\right\rangle\\
&-\alpha\left|\left|\left(X_t^{i,N}-X_t^{j,N}\right)-\left(X_t^i-X_t^j\right)\right|\right|^2\\
\geq&-\alpha\left|\left|\left(X_t^{i,N}-X_t^{j,N}\right)-\left(X_t^i-X_t^j\right)\right|\right|^2\\
\geq&-2\alpha\left\{\left|\left|X_t^{i,N}-X_t^i\right|\right|^2+\left|\left|X_t^{j,N}-X_t^j\right|\right|^2\right\}\,.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\mathbb{E}\left\{-\frac{2}{N}\sum_{i=1}^N\sum_{j=1}^N\Delta_2(i,j,t)\right\}&=\frac{1}{2}\mathbb{E}\left\{-\frac{2}{N}\sum_{1\leq i,j\leq N}^N\Big(\Delta_2(i,j,t)+\Delta_2(j,i,t)\Big)\right\}\\
&\leq 4\alpha\sum_{i=1}^N\left|\left|X_t^{i,N}-X_t^i\right|\right|^2\,. \qquad (19)
\end{aligned}
$$

By definition of θ in Assumption (A-3), for any $x,y\in\mathbb{R}^d$ we have the inequality $\langle\nabla V(x)-\nabla V(y)\,;\,x-y\rangle\geq-\theta\,||x-y||^2$. This implies

$$
-2\sum_{i=1}^N\Delta_1(i,t)\leq 2\theta\sum_{i=1}^N\xi_i(t)\,. \qquad (20)
$$

We now deal with the sum containing $\Delta_3(i,j,t)$. We apply Cauchy-Schwarz inequality:

$$
\begin{aligned}
-\mathbb{E}\left[\sum_{j=1}^N\Delta_3(i,j,t)\right]&\leq\left\{\mathbb{E}\left[\left|\left|X_t^{i,N}-X_t^i\right|\right|^2\right]\right\}^{\frac{1}{2}}\left\{\sum_{j=1}^N\sum_{k=1}^N\mathbb{E}\left[\left\langle\rho_j^i(t)\,;\,\rho_k^i(t)\right\rangle\right]\right\}^{\frac{1}{2}}\\
&\text{with}\quad \rho_j^i(t):=\nabla F(X_t^i-X_t^j)-\nabla F*\mu_t\left(X_t^i\right)\,.
\end{aligned}
$$

Let us prove that $\mathbb{E}\left[\left\langle\rho_j^i(t)\,;\,\rho_k^i(t)\right\rangle\right]=0$ for any $j\neq k$. We use the following conditioning:

$$
\mathbb{E}\left[\left\langle\rho_j^i(t)\,;\,\rho_k^i(t)\right\rangle\right]=\mathbb{E}\left\{\mathbb{E}\left[\left\langle\rho_j^i(t)\,;\,\rho_k^i(t)\right\rangle\;\middle|\;X_0\right]\right\}\,.
$$

We now condition by $\left(B_s^i\right)_{0\leq s\leq t}$. However, we have

$$
\mathbb{E}\Big(\nabla F\left(X_t^i-X_t^k\right)\;\big|\;\left(B_s^i\right)_{0\leq s\leq t},X_0\Big)=\nabla F*\mu_t\left(X_t^i\right)\,.
$$

We deduce immediatly:

$$\mathbb{E}\left(\rho_k^i(t)\mid\left(B_s^i\right)_{0\le s\le t},X_0\right)=0\,,$$

so that

$$\mathbb{E}\left\{\langle\rho_j^i(t)\,;\,\rho_k^i(t)\rangle\right\}=0\,,$$

for any $j\neq k$. And, if $j=k$, we have

$$\mathbb{E}\left\{\left|\left|\rho_j^i(t)\right|\right|^2\right\}=\mathbb{E}\left\{\left|\left|\nabla F\left(X_t^i-X_t^j\right)-\nabla F*\mu_t\left(X_t^i\right)\right|\right|^2\right\}\,.$$

The diffusions X^i and X^j are not independent but they are independent conditionally to the initial random variables. However, according to Hypothesis (A-6), we have $F(x)=G\left(||x||\right)$ where G is a polynomial function of degree $2n$, we have the following inequality:

$$\mathbb{E}\left[||\nabla F(X_t-Y_t)-\nabla F*\mu_t(X_t)||^2\right]\le C\left(1+\mathbb{E}\left[||X_t||^{4n-2}\right]\right)\,,$$

X_t and Y_t being two independent random variables with common law μ_t and C is a positive constant. Then, we use the control of the moments obtained in [HIP08, Theorem 2.13] and we obtain the following majoration:

$$\sup_{t\ge0}\mathbb{E}\left[||\nabla F(X_t-Y_t)-\nabla F*\mu_t(X_t)||^2\right]\le C(\mu_0)\,,$$

$C(\mu_0)$ being a function of the $8q^2$ moment of the law μ_0. Consequently, we have

$$\mathbb{E}\left\{\left|\left|\rho_j^i(t)\right|\right|^2\mid X_0\right\}\le C(\mu_0)\,,$$

for any $1\le i,j\le N$. By taking the expectation, we obtain

$$\mathbb{E}\left\{\left|\left|\rho_j^i(t)\right|\right|^2\right\}\le C(\mu_0)\,,$$

for any $1\le i,j\le N$. Therefore, we deduce the following inequality:

$$-\mathbb{E}\left[\sum_{j=1}^N\Delta_3(i,j,t)\right]\quad\le\sqrt{C(\mu_0)}\sqrt{N\mathbb{E}\left[\xi_i(t)\right]}\,.\tag{21}$$

By combining (19), (20) and (21), we obtain

$$\frac{d}{dt}\sum_{i=1}^N\mathbb{E}\left[\xi_i(t)\right]\le2\sum_{i=1}^N\left\{(\theta+2\alpha)\,\mathbb{E}\left[\xi_i(t)\right]+\frac{\sqrt{C(\mu_0)}}{\sqrt{N}}\sqrt{\mathbb{E}\left[\xi_i(t)\right]}\right\}\,.\tag{22}$$

However, the particles are exhangeables. Consequently, for any $1\le i\le N$, we have

$$\frac{d}{dt}\mathbb{E}\left\{\xi_i(t)\right\}\le2\left(\theta+2\alpha\right)\mathbb{E}\left\{\xi_i(t)\right\}+\frac{2\sqrt{C(\mu_0)}}{\sqrt{N}}\sqrt{\mathbb{E}\left[\xi_i(t)\right]}\,.$$

By introducing $\tau_i(t) := \sqrt{\mathbb{E}\left\{\xi_i(t)\right\}}$, we obtain

$$\tau_i'(t) \le (\theta + 2\alpha)\left\{\tau_i(t) + \frac{\sqrt{C(\mu_0)}}{(\theta + 2\alpha)\sqrt{N}}\right\}$$

The application of Grönwall lemma yields

$$\mathbb{E}\left\{\left|\left|X_t^{i,N} - X_t^i\right|\right|^2\right\} \le \frac{C(\mu_0)}{N\left(\theta + 2\alpha\right)^2}\exp\left[2\left(\theta + 2\alpha\right)t\right] \,.$$

We obtain (18) by taking the supremum for t running between 0 and T. $\square$

Consequently, there exist two positive constants K, C such that, under the set of Assumptions (A), we have the coupling result:

$$\sup_{0\le t\le T}\mathbb{E}\left\{\left|\left|X_t^{i,N} - X_t^i\right|\right|^2\right\} \le K^2\frac{e^{2CT}}{N}\,, \tag{23}$$

This result holds for any $T > 0$.

3.1 Decorrelation for two particles

We now are able to provide the proof of Theorem 1.3, that we remind the reader.

Theorem 3.2. *Let f_1 and f_2 be two Lipschitz-continuous functions. Under the set of Assumptions (A), for all $\epsilon > 0$, for all $T > 0$, there exist $t_0(\epsilon)$ and $N_0(\epsilon)$ such that*

$$\sup_{N\ge N_0(\epsilon)}\ \sup_{t\in[t_0(\epsilon);t_0(\epsilon)+T]}\left|\mathrm{Cov}\left[f_1\left(X_t^{1,N}\right)\,;\,f_2\left(X_t^{2,N}\right)\right]\right| \le \epsilon\,.$$

Proof. Let T and ϵ be any positive reals. Set $0 \le t$.

Step 1. We use the following decomposition

$$\begin{aligned}
&\mathrm{Cov}\left(f_1\left(X_t^{1,N}\right)\,;\,f_2\left(X_t^{2,N}\right)\right)\\
=&\mathbb{E}\left\{f_1\left(X_t^{1,N}\right)\left[f_2\left(X_t^{2,N}\right) - f_2\left(X_t^2\right)\right]\right\}\\
&+\mathbb{E}\left\{f_2\left(X_t^2\right)\left[f_1\left(X_t^{1,N}\right) - f_1\left(X_t^1\right)\right]\right\}\\
&+\mathrm{Cov}\left(f_1\left(X_t^1\right)\,;\,f_2\left(X_t^2\right)\right)\\
&+\mathbb{E}\left\{f_1\left(X_t^1\right)\right\}\left[\mathbb{E}\left(f_2\left(X_t^2\right)\right) - \mathbb{E}\left(f_2\left(X_t^{2,N}\right)\right)\right]\\
&+\mathbb{E}\left\{f_2\left(X_t^{2,N}\right)\right\}\left[\mathbb{E}\left(f_1\left(X_t^1\right)\right) - \mathbb{E}\left(f_1\left(X_t^{1,N}\right)\right)\right]\\
=:&T_1 + T_2 + T_3 + T_4 + T_5\,.
\end{aligned}$$

Step 2. We can control T_1 in the following way.

$$|T_1| \le \sqrt{\mathbb{E}\left\{\left|\left|f_1\left(X_t^{1,N}\right)\right|\right|^2\right\}}\sqrt{\mathbb{E}\left\{\left|\left|f_2\left(X_t^{2,N}\right) - f_2\left(X_t^2\right)\right|\right|^2\right\}}$$

by Cauchy-Schwarz inequality. The triangular inequality provides us:

$$\left|\left|f_1\left(X_t^{1,N}\right)\right|\right|^2 \leq 3\left|\left|f_1(0)\right|\right|^2 + 3\left|\left|f_1\left(X_t^1\right) - f_1(0)\right|\right|^2 + 3\left|\left|f_1\left(X_t^{1,N}\right) - f_1\left(X_t^1\right)\right|\right|^2$$

Since f_1 is a Lipschitz-continuous function, there exists $\rho > 0$ such that

$$\left|\left|f_1\left(X_t^{1,N}\right)\right|\right|^2 \leq 3\left|\left|f_1(0)\right|\right|^2 + 3\rho^2\left|\left|X_t^1\right|\right|^2 + 3\rho^2\left|\left|X_t^{1,N} - X_t^1\right|\right|^2 .$$

Due to the inequalities (14) and (23), we have

$$\left|\left|f_1\left(X_t^{1,N}\right)\right|\right|^2 \leq 3\rho^2\left(M_0 + \left|\left|f_1(0)\right|\right|^2 + K^2\frac{e^{2Ct}}{N}\right)$$

Still by using the coupling result (23), we have

$$\mathbb{E}\left\{\left|\left|f_2\left(X_t^{2,N}\right) - f_2\left(X_t^2\right)\right|\right|^2\right\} \leq \rho^2 K^2 \frac{e^{2Ct}}{N} ,$$

so that the term T_1 is bounded like so

$$|T_1| \leq 2\rho\sqrt{M_0 + \left|\left|f_1(0)\right|\right|^2 + K^2\frac{e^{2Ct}}{N}}\,\rho K\frac{e^{Ct}}{\sqrt{N}} .$$

We deduce that we have

$$|T_1| \leq 2\rho^2 K\sqrt{K_0 + K^2\frac{e^{2Ct}}{N}}\,\frac{e^{Ct}}{\sqrt{N}} .$$

Step 3. By proceeding similarly, we have the following boundedness of the fifth term.

$$|T_5| \leq 2\rho^2 K\sqrt{K_0 + K^2\frac{e^{2Ct}}{N}}\,\frac{e^{Ct}}{\sqrt{N}} .$$

Step 4. By using the uniform boundedness of the moments (14), Jensen's inequality and the coupling result (23), we obtain the following control:

$$\max\left\{|T_2|\,;|T_4|\right\} \leq \rho^2 K\sqrt{M_0}\,\frac{e^{Ct}}{\sqrt{N}} .$$

Step 5. Finally, the limit (17) provides us the existence of a decreasing function φ which limit at infinity is 0 such that

$$|T_3| \leq \varphi(t) .$$

Step 6. The triangular inequality gives us

$$\left|\mathrm{Cov}\left(f_1\left(X_t^{1,N}\right)\,;\,f_2\left(X_t^{2,N}\right)\right)\right| \leq |T_1| + |T_2| + |T_3| + |T_4| + |T_5| .$$

Let $t_0(\epsilon)$ be a positive real such that $\varphi(t_0(\epsilon)) < \frac{\epsilon}{2}$. Then, we take $N_0(\epsilon)$ large enough such that we have $|T_1| + |T_2| + |T_4| + |T_5| \leq \frac{\epsilon}{2}$. We deduce that for any $t \in [t_0(\epsilon); t_0(\epsilon) + T]$, for any $N \geq N_0(\epsilon)$, we have

$$\left|\text{Cov}\left[f_1\left(X_t^{1,N}\right) ; f_2\left(X_t^{2,N}\right)\right]\right| \leq \epsilon .$$

□

This theorem means that, for a time and a number of particles sufficiently large, two particles are as independent as we desire. Moreover, the convergence in time is exponential.
We do not need neither V nor F to be convex. Nevertheless, if both potentials V and F are convex, we know that we have a uniform coupling between the particles and the inequality (23) becomes

$$\sup_{t\geq 0}\mathbb{E}\left\{\left|\left|X_t^{i,N} - X_t^i\right|\right|\right\} \leq \frac{K(\mu_0)^2}{N}, \tag{24}$$

so that the four terms T_1, T_2, T_4 and T_5 (defined in the proof of Theorem 1.3) are bounded like so

$$\sup_{t\geq 0}\max\left\{|T_1|; |T_2|; |T_4|; |T_5|\right\} \leq \frac{\lambda}{4\sqrt{N}} .$$

In the previous (uniform) inequality, λ is a positive constant. Immediately, we have the majoration:

$$\left|\text{Cov}\left(f_1\left(X_t^{1,N}\right) ; f_2\left(X_t^{2,N}\right)\right)\right| \leq \varphi(t) + \lambda\frac{1}{\sqrt{N}} .$$

Taking t and N sufficiently large yields

$$\left|\text{Cov}\left(f_1\left(X_t^{1,N}\right) ; f_2\left(X_t^{2,N}\right)\right)\right| \leq \epsilon ,$$

since the function φ is decreasing. This ends the proof of Theorem 1.4, that we remind the reader.

Theorem 3.3. *Let f_1 and f_2 be two Lipschitz-continuous functions. Under the set of assumptions (A), if moreover, V and F are convex then, for all $\epsilon > 0$, there exist $t_0(\epsilon)$ and $N_0(\epsilon)$ such that*

$$\sup_{N\geq N_0(\epsilon)}\sup_{t\geq t_0(\epsilon)}\left|\text{Cov}\left[f_1\left(X_t^{1,N}\right) ; f_2\left(X_t^{2,N}\right)\right]\right| \leq \epsilon .$$

In this paragraph, we only look at two particles for the sake of simplicity.

Remark 3.4. *In fact, by proceeding similarly, we can obtain a more general result that we now present. Let $f_1, \cdots, f_k$ be k Lipschitz-continuous functions.*

Then, under the set of assumptions (A), for all $\epsilon > 0$, for all $T > 0$, there exist $t_0(\epsilon)$ and $N_0(\epsilon)$ such that

$$\sup_{N\geq N_0(\epsilon)}\sup_{t\in[t_0(\epsilon);t_0(\epsilon)+T]}\left|\mathbb{E}\left\{\prod_{i=1}^k f_i\left(X_t^{i,N}\right)\right\}-\prod_{i=1}^k\mathbb{E}\left\{f_i\left(X_t^{i,N}\right)\right\}\right|\leq\epsilon\,.$$

If moreover, both confining and interacting potentials V and F are convex, we have

$$\sup_{N\geq N_0(\epsilon)}\sup_{t\geq t_0(\epsilon)}\left|\mathbb{E}\left\{\prod_{i=1}^k f_i\left(X_t^{i,N}\right)\right\}-\prod_{i=1}^k\mathbb{E}\left\{f_i\left(X_t^{i,N}\right)\right\}\right|\leq\epsilon\,,$$

for any $\epsilon > 0$.

In [DMT13], we provide a result of propagation of chaos which is uniform with respect to the time without assuming the convexity of V nor the one of F. However, we can not use these results in our study since the results in [DMT13] are with Wasserstein distance, $\mathbb{W}_2$. In other words, these results hold with diffusions X^1 and $X^{1,N}$ which are not necessarily defined with the same Brownian motions. Consequently, we can not obtain the uniform coupling result between the particles in the mean-field system and their hydrodynamical limit if V or F are not convex.

3.2 Creation of chaos for the empirical measure

When the initial random variables $X_0^1,\cdots,X_0^N$ are independent, the empirical measure $\eta_t^N := \frac{1}{N}\sum_{j=1}^N\delta_{X_t^{j,N}}$ converges as N goes to infinity toward the deterministic measure μ_t (the law at time t of the McKean-Vlasov diffusion). In particular, for any Lipschitz-continuous function f, the quantity $\eta_t^N(f) := \int_{\mathbb{R}} f(x)\eta_t^N(dx)$ converges toward the deterministic value $\int_{\mathbb{R}} f(x)\mu_t(dx)$ so that the covariance of $\eta_t^N(f_1)$ and $\eta_t^N(f_2)$ goes to 0 where f_1 and f_2 are two Lipschitz-continuous functions.
However, due to Theorem 1.3, we have the Theorem 1.6, that we remind the reader.

Theorem 3.5. *Let f_1 and f_2 be two Lipschitz-continuous functions. Under the set of assumptions (A), for all $\epsilon > 0$, for all $T > 0$, there exists $t_1(\epsilon)$ and $N_1(\epsilon)$ such that*

$$\sup_{N\geq N_1(\epsilon)}\sup_{t\in[t_1(\epsilon);t_1(\epsilon)+T]}\left|\mathrm{Cov}\left[\eta_t^N(f_1)\,;\,\eta_t^N(f_2)\right]\right|\leq\epsilon$$

with $\eta_t^N(f) := \frac{1}{N}\sum_{i=1}^N f\left(X_t^{i,N}\right)$. If, moreover, both V and F are convex, we have

$$\sup_{N\geq N_1(\epsilon)}\sup_{t\geq t_1(\epsilon)}\left|\mathrm{Cov}\left[\eta_t^N(f_1)\,;\,\eta_t^N(f_2)\right]\right|\leq\epsilon\,,$$

for any $\epsilon > 0$.

Proof. By definition of $\eta_t^N(f)$, we have

$$\mathrm{Cov}\left(\eta_t^N(f_1)\,;\,\eta_t^N(f_2)\right) = \frac{1}{N^2}\sum_{1\le i,j\le N}\mathrm{Cov}\left(f_1\left(X_t^{i,N}\right)\,;\,f_2\left(X_t^{j,N}\right)\right)\,.$$

Consequently, if $N \ge N_0(\epsilon)$ (where the integer $N_0(\epsilon)$ has been defined in Theorem 1.3), we have by triangular inequality:

$$\begin{aligned}&\sup_{t\in[t_0(\epsilon);t_0(\epsilon)+T]}\left|\mathrm{Cov}\left(\eta_t^N(f_1)\,;\,\eta_t^N(f_2)\right)\right|\\ &\le\left(1-\frac{1}{N_0(\epsilon)}\right)\epsilon+\frac{1}{N^2}\sum_{i=1}^N\left|\mathrm{Cov}\left(f_1\left(X_t^{i,N}\right)\,;\,f_2\left(X_t^{i,N}\right)\right)\right|\,.\end{aligned}$$

Nevertheless, due to the hypotheses, we have the convergence of the quantity

$$\mathrm{Cov}\left(f_1\left(X_t^i\right)\,;\,f_2\left(X_t^i\right)\right)$$

to

$$\int_{\mathbb{R}}f_1(x)f_2(x)\mu(dx)-\left(\int_{\mathbb{R}}f_1(x)\mu(dx)\right)\left(\int_{\mathbb{R}}f_2(x)\mu(dx)\right)\,,$$

as t goes to infinity. Then, since f_1 and f_2 are Lipschitz-continuous functions, thanks to the coupling inequality (23), we obtain that for all $\epsilon > 0$, the quantity

$$\left|\mathrm{Cov}\left(f_1\left(X_t^{i,N}\right)\,;\,f_2\left(X_t^{i,N}\right)\right)-\left[\int_{\mathbb{R}}f_1f_2\mu-\left(\int_{\mathbb{R}}f_1\mu\right)\left(\int_{\mathbb{R}}f_2\mu\right)\right]\right|$$

is less than ϵ if t and N are large enough. Particularly, we deduce the boundedness of $\left|\mathrm{Cov}\left(f_1\left(X_t^{i,N}\right)\,;\,f_2\left(X_t^{i,N}\right)\right)\right|$:

$$\sup_{N\ge1}\sup_{t\ge0}\left|\mathrm{Cov}\left(f_1\left(X_t^{i,N}\right)\,;\,f_2\left(X_t^{i,N}\right)\right)\right|\le M\,,$$

M being a positive constant.
Taking $N_1(\epsilon) := \max\left\{N_0(\epsilon)\,;\,\frac{MN_0(\epsilon)}{\epsilon}\right\}$ yields

$$\sup_{t\in[t_0(\epsilon);t_0(\epsilon)+T]}\left|\mathrm{Cov}\left(\eta_t^N(f_1)\,;\,\eta_t^N(f_2)\right)\right|\le\epsilon$$

The second part of the theorem can be proved in a similar way so it is left to the reader. □

This result can be interpreted in the following way. The larger both N and t are, the more deterministic is the empirical measure η_t^N.

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